

SPHERICAL AND HYPERBOLIC LENGTHS OF IMAGES OF ARCS

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Abstract

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function on the unit disc which is in the Dirichlet class, so the Euclidean area of the image, counting multiplicity, is finite. The Euclidean length of a radial arc of hyperbolic length ρ is then $o(\rho^{1/2})$. In this note we consider the corresponding results when f maps into the unit disc with the hyperbolic metric or the Riemann sphere with the spherical metric. Similar but not identical results hold.

1. Introduction

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function on the unit disc \mathbb{D} which is in the Dirichlet class, so the Euclidean area of the image $f(\mathbb{D})$, counting multiplicity, is finite. Keogh [K] showed that a radial arc $[0, z]$ in the disc is mapped to an arc with Euclidean length \mathcal{E} that satisfies

$$\mathcal{E} = o(\rho(0, z)^{1/2})$$

where ρ is the hyperbolic length in \mathbb{D} . The paper [BC] explored the result in some detail. In this paper we will consider the corresponding results when the image domain is either the disc or the extended complex plane. For these we will use the natural Riemannian metrics: the hyperbolic metric on the unit disc and the spherical metric on the extended complex plane. Similar results hold in these cases, essentially because we can localise the result to a small disc with an Euclidean image. However, there are interesting differences and the arguments make the importance of the hyperbolic metric still more apparent.

We will consider domains A with a Riemannian metric $ds = \lambda_A(z)|dz|$. Here λ_A is a strictly positive function on A giving the density of the metric. If $f : A \rightarrow B$ is an analytic map, then the derivative has norm

$$\|f'(z_o)\|_{A \rightarrow B} = |f'(z_o)| \frac{\lambda_B(f(z_o))}{\lambda_A(z_o)}.$$

This is the factor by which f changes infinitesimal lengths at the point z_o for the metrics on A and B . The area is changed by the square of this factor.

On the complex plane \mathbb{C} we will use the Euclidean metric $|dz|$ with density 1. On the unit disc \mathbb{D} we will use the hyperbolic metric with density

$$\lambda_H(z) = \frac{2}{1 - |z|^2}.$$

This has constant curvature -1 . On the extended complex plane we will use the spherical metric with density

$$\lambda_S(z) = \frac{2}{1 + |z|^2}.$$

This has constant curvature $+1$ and is isometric, under stereographic projection, with the unit sphere in Euclidean \mathbb{R}^3 . This is the Riemann sphere and we will denote it by \mathbb{P} . The subscripts H, E, S will be used to specify the hyperbolic, Euclidean and spherical metrics respectively. So d_H, d_E, d_S are the distances on these three spaces and $\mathbb{A}_H, \mathbb{A}_E, \mathbb{A}_S$ are the area measures.

Let $f : \mathbb{D} \rightarrow B$ be an analytic map into some domain B with density λ . The area of the image, counting multiplicity is

$$\int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 \lambda(f(re^{i\theta}))^2 d\theta r dr = \int_{\mathbb{D}} \|f'(z)\|_{E \rightarrow E}^2 d\mathbb{A}_E(z).$$

We will abbreviate this to $\mathbb{A}_B(f(\mathbb{D}))$ when there is no chance of confusion. It is more natural to write this as an area integral over the unit disc using the hyperbolic metric:

$$\mathbb{A}_B(f(\mathbb{D})) = \int_{\mathbb{D}} \|f'(z)\|_{H \rightarrow B}^2 d\mathbb{A}_H(z) .$$

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic map with $\mathbb{A}_E(f(\mathbb{D})) < \infty$, so f is in the Dirichlet class. In [BC] it was shown that

$$\|f'(z)\|_{H \rightarrow E} \leq \left(\frac{\mathbb{A}_E(f(\mathbb{D}))}{4\pi} \right)^{1/2} . \quad (1.1)$$

Choose a direction from the origin and consider a radial arc of hyperbolic length ρ from 0 in this direction. The image of this arc will have Euclidean length $L_E(\rho)$. By integrating (1.1) along the radial arc we can show that

$$L_E(\rho) = o(\rho^{1/2}) \quad (1.2)$$

as $\rho \nearrow \infty$, that is as the radial arc extends to the boundary.

The purpose of this note is to consider the corresponding results for maps $f : \mathbb{D} \rightarrow \mathbb{D}$ or $f : \mathbb{D} \rightarrow \mathbb{P}$. For the hyperbolic image, both (1.1) and (1.2) hold. In the spherical case, (1.1) fails when the area $\mathbb{A}_S(f(\mathbb{D}))$ is large. However, it does hold when this area is small and this is sufficient to establish (1.2).

In all cases, the arguments are very similar in spirit to those in [BC] and so they are not laboured. That paper considers many analogues and extensions of the results and these, similarly, can be established for hyperbolic and spherical images.

2. The norm of the derivative

We wish to establish bounds on the derivative $\|f'(z)\|_{H \rightarrow B}$ in terms of the area $\mathbb{A}_B(f(\mathbb{D}))$. In the Euclidean case this is very simple and is already done in Keogh's paper [K]. We include a proof to compare with later results for the hyperbolic and spherical images.

Proposition 2.1

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic map with the Euclidean area of the image $\mathbb{A}_E(f(\mathbb{D}))$, counting multiplicity, finite. Then

$$\|f'(z_o)\|_{H \rightarrow E} \leq \left(\frac{\mathbb{A}_E(f(\mathbb{D}))}{4\pi} \right)^{1/2} .$$

Proof:

Composing f with a hyperbolic isometry will not alter the area of the image. Hence we can assume that $z_o = 0$. Let $f(z) = \sum a_n z^n$ be the power series for f . Then $\|f'(0)\|_{H \rightarrow E} = |f'(0)|/2 = |a_1|/2$. The usual integration of the corresponding Fourier series gives

$$\mathbb{A}_E(f(\mathbb{D})) = \pi \sum_{n=1}^{\infty} n |a_n|^2 .$$

So we certainly have

$$\|f'(0)\|_{H \rightarrow E}^2 = \frac{|a_1|^2}{4} \leq \frac{\mathbb{A}_E(f(\mathbb{D}))}{4\pi}$$

as required. The inclusion map $f : \mathbb{D} \hookrightarrow \mathbb{C}$ shows that the constant $1/4\pi$ in this result is the best possible. \square

We can also apply this result to analytic maps $f : \Delta \rightarrow \mathbb{C}$ defined on any hyperbolic domain Δ . In particular, let Δ be the ball of hyperbolic radius ρ_o about z_o :

$$\Delta = B(z_o, \rho_o) = \{z \in \mathbb{D} : d_H(z_o, z) < \rho_o\} .$$

Let $r_o = \tanh \frac{1}{2}\rho_o$ (so the disc of hyperbolic radius ρ_o centred on 0 has Euclidean radius r_o). Then the Möbius transformation

$$T : z \mapsto \frac{r_o z + z_o}{1 + \bar{z}_o r_o z}$$

maps the unit disc conformally onto Δ . So we can apply the Proposition to $g = f \circ T$ to obtain

$$r_o \|f'(z_o)\|_{H \rightarrow E} \leq \left(\frac{\mathbb{A}_E(f(\Delta))}{4\pi} \right)^{1/2}.$$

This gives us a way of localising the result and hence of applying it when the image domain has a different metric.

Proposition 2.2

Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map with the hyperbolic area of the image $\mathbb{A}_H(f(\mathbb{D}))$, counting multiplicity, finite. Then

$$\|f'(z_o)\|_{H \rightarrow H} \leq \left(\frac{\mathbb{A}_H(f(\mathbb{D}))}{4\pi} \right)^{1/2}.$$

Proof:

By composing with hyperbolic isometries, we may assume that $z_o = 0$ and $f(z_o) = 0$. Then $\|f'(z_o)\|_{H \rightarrow H} = |f'(0)|$.

The hyperbolic density in \mathbb{D} is always at least 2, so $\mathbb{A}_E(f(\Delta)) \leq \frac{1}{4}\mathbb{A}_H(f(\Delta))$. Therefore, Proposition 2.1 shows that

$$\|f'(0)\|_{H \rightarrow H} = 2\|f'(0)\|_{H \rightarrow E} \leq 2 \left(\frac{\mathbb{A}_E(f(\mathbb{D}))}{4\pi} \right)^{1/2} \leq \left(\frac{\mathbb{A}_H(f(\mathbb{D}))}{4\pi} \right)^{1/2}$$

as required. \square

For the map $f : z \mapsto \varepsilon z$ we have $\|f'(0)\|_{H \rightarrow H} = \varepsilon$ while $\mathbb{A}_H(f(\mathbb{D})) = 4\pi\varepsilon^2/(1 - \varepsilon^2)$. So we see that the constant $1/4\pi$ in the proposition is the best possible.

The corresponding result for the spherical metric fails. For consider a univalent map $k : \mathbb{D} \rightarrow \mathbb{P}$ with $k(0) = 0$ and whose image has spherical area $4\pi = \mathbb{A}_S(\mathbb{P})$. For example, the Koebe function $k : z \mapsto z/(1 - z)^2$. This has $\mathbb{A}_S(k(\mathbb{D})) = 4\pi$ and

$$\|k'(0)\|_{H \rightarrow S} = |k'(0)| \neq 0.$$

For any $\lambda \neq 0$, the map $f(z) = \lambda k(z)$ also has $\mathbb{A}_S(k(\mathbb{D})) = 4\pi$ but $\|f'(0)\|_{H \rightarrow S} = |\lambda|$. So we can not have any inequality of the form

$$\|f'(0)\|_{H \rightarrow S}^2 \leq c\mathbb{A}_S(f(\mathbb{D})).$$

Nonetheless, the result does hold provided that the spherical area of the image is sufficiently small.

Proposition 2.3

Let $f : \mathbb{D} \rightarrow \mathbb{P}$ be an analytic map with the spherical area of the image $\mathbb{A}_S(f(\mathbb{D}))$, counting multiplicity, less than 2π . Then

$$\|f'(z_o)\|_{H \rightarrow S} \leq c\mathbb{A}_E(f(\mathbb{D}))^{1/2}$$

for some constant c .

Proof:

We may assume that $z_o = 0$ and $f(z_o) = 0$. So $\|f'(0)\|_{H \rightarrow S} = |f'(0)|$.

Choose $\delta > 0$ so that the spherical area of the ball

$$B_S(c, \delta) = \{z \in \mathbb{P} : d_S(c, z) < \delta\}$$

is less than $\frac{1}{2}\pi$. For this δ , find a maximal set of points $c_0, c_1, c_2, \dots, c_K$ with $c_0 = 0$, the other points c_1, c_2, c_K in $\mathbb{P} \setminus f(\mathbb{D})$, and all of the distances $d_S(c_i, c_j) \geq \delta$ for $i \neq j$. Since K is maximal, there can be no point of $\mathbb{P} \setminus f(\mathbb{D})$ lying outside the balls $B_S(c_j, \delta)$. Hence these $K + 1$ balls cover all of $\mathbb{P} \setminus f(\mathbb{D})$ and so

$$(K + 1)\mathbb{A}_S(B_S(0, \delta)) \geq \mathbb{A}_S(\mathbb{P} \setminus f(\mathbb{D})) \geq 2\pi.$$

This implies that

$$K + 1 \geq \frac{2\pi}{\mathbb{A}_S(B(0, \delta))} > 4.$$

So we can find 3 points w_0, w_1, w_∞ in $\mathbb{P} \setminus f(\mathbb{D})$ which satisfy

$$d_S(0, w_i) > \delta, \quad d_S(w_i, w_j) > \delta \quad (2.1)$$

for $i \neq j$. We can now compare the spherical metric on \mathbb{P} with the hyperbolic metric on $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$.

The three punctured sphere $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$ has a hyperbolic metric and the conditions (2.1) allow us to estimate its properties uniformly. From now on, consider $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$ with this metric. This hyperbolic metric can be written as $\lambda(z)ds_S(z)$ where ds_S denotes the infinitesimal spherical metric and λ is the density relative to this spherical metric. The function λ is bounded away from 0 and tends to $+\infty$ at the three punctures. The conditions (2.1) imply that there are constant $K, K', K'' > 0$, depending only on δ , with the following properties:

- (a) $\lambda(0) \geq K$;
- (b) the spherical balls $B_S(w_i, K')$ are disjoint and at least hyperbolic distance 1 from 0 in $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$;
- (c) The hyperbolic density $\lambda(z) \leq K''$ for all points z within a hyperbolic distance 1 from 0 in $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$.

(Compare this with the estimates in [A], 1-9. There is a Möbius transformation $T : \mathbb{P} \rightarrow \mathbb{P}$ that maps our three punctured sphere $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$ onto the standard three punctured sphere $\mathbb{P} \setminus \{0, 1, \infty\}$. The conditions (2.1) show that this transformation only distorts the metrics by a controlled amount.)

Now $f : \mathbb{D} \rightarrow \mathbb{P} \setminus \{w_0, w_1, w_\infty\}$ is an analytic map between two hyperbolic domains so the Schwarz – Pick lemma implies that f is a contraction for the hyperbolic metrics. This implies that f maps the hyperbolic disc $\Delta = B_H(0, 1)$ with hyperbolic radius 1 into the region

$$\{w \in \mathbb{P} \setminus \{w_0, w_1, w_\infty\} : d_H(0, w) < 1\}.$$

Condition (a) above shows that $\|f'(0)\|_{H \rightarrow S} \leq \frac{2}{K}\|f'(0)\|_{H \rightarrow H}$. Condition (c) shows that the hyperbolic area $\mathbb{A}_H(f(\Delta))$ is at most K''^2 times the spherical area $\mathbb{A}_S(f(\Delta))$. Finally, we can apply Proposition 2.2 to the map $f|_\Delta : \Delta \rightarrow \mathbb{P} \setminus \{w_0, w_1, w_\infty\}$ (or, more properly, to its lift to the universal cover of $\mathbb{P} \setminus \{w_0, w_1, w_\infty\}$). This gives

$$r_o \|f'(0)\|_{H \rightarrow H} \leq \left(\frac{\mathbb{A}_H(f(\Delta))}{4\pi} \right)^{1/2} \leq \left(\frac{K''^2 \mathbb{A}_S(f(\Delta))}{4\pi} \right)^{1/2}$$

for $r_o = \tanh \frac{1}{2}$. Putting all of these together gives

$$\|f'(0)\|_{H \rightarrow S} \leq \frac{2K''}{r_o K} \left(\frac{\mathbb{A}_S(f(\mathbb{D}))}{4\pi} \right)^{1/2}$$

as required. \square

The argument used in the proof shows that, for any $C < \mathbb{A}_S(\mathbb{P}) = 4\pi$, there is a constant $K(C)$ with

$$\|f'(z_o)\|_{H \rightarrow S} \leq K(C) \mathbb{A}_S(f(\mathbb{D}))^{1/2}$$

provided that $\mathbb{A}_S(f(\mathbb{D})) \leq C$.

3. The lengths of image arcs

We will need to introduce some notation that will apply throughout the remainder of the paper.

Let $f : \mathbb{D} \rightarrow B$ be an analytic map into one of the domains $B = \mathbb{D}, \mathbb{C}, \mathbb{P}$. Let $\gamma : [0, \infty) \rightarrow \mathbb{D}$ be a radial hyperbolic geodesic with unit speed and argument θ , so $\gamma(t) = re^{i\theta}$ where $r = \tanh \frac{1}{2}t$. Consider the arc $\gamma[0, \rho_o]$ with hyperbolic length ρ_o . This has an image of length

$$L_B(\rho_o) = \int_0^{\rho_o} \|f'(\gamma(t))\|_{H \rightarrow B} dt$$

in the metric for B .

For each $\rho < \infty$, the image of the hyperbolic ball $B_H(0, \rho) = \{z \in \mathbb{D} : d_H(0, z) < \rho\}$ has finite area, which we denote by $A(\rho) = \mathbb{A}_B(f(\Delta))$. Then $A(\rho)$ is an increasing function. If the area of the entire image $\mathbb{A}_B(f(\mathbb{D}))$ is finite, then $A(\rho)$ is bounded above and converges to $\mathbb{A}_B(f(\mathbb{D}))$ as $\rho \nearrow \infty$.

We wish to apply the propositions of §2 to the function f restricted to hyperbolic discs $B_H(\gamma(t), \delta)$ for some hyperbolic radius δ . This disc lies inside the ball $B_H(0, t + \delta)$ and outside the disc $B_H(0, t - \delta)$. So the area $\mathbb{A}_B(f(\Delta)) \leq A(t + \delta) - A(t - \delta)$.

Initially, let us consider the Euclidean case $f : \mathbb{D} \rightarrow \mathbb{C}$. For this Proposition 2.1 gives us

$$\|f'(\gamma(t))\|_{H \rightarrow E} \leq \frac{1}{\tanh \frac{1}{2}\delta} \left(\frac{A(t + \delta) - A(t - \delta)}{4\pi} \right)^{1/2}.$$

So integrating and applying the Cauchy – Schwarz inequality gives

$$\begin{aligned} L_E(\rho_o) &= \int_0^{\rho_o} \|f'(\gamma(t))\|_{H \rightarrow E} dt \\ &\leq \int_0^{\rho_o} \frac{1}{\tanh \frac{1}{2}\delta} \left(\frac{A(t + \delta) - A(t - \delta)}{4\pi} \right)^{1/2} dt \\ &\leq \left(\int_0^{\rho_o} \frac{1}{4\pi \tanh^2 \frac{1}{2}\delta} (A(t + \delta) - A(t - \delta)) dt \right)^{1/2} \left(\int_0^{\rho_o} 1 dt \right)^{1/2} \\ &\leq \left(\int_0^{\rho_o} \frac{1}{4\pi \tanh^2 \frac{1}{2}\delta} (A(t + \delta) - A(t - \delta)) dt \right)^{1/2} \rho_o^{1/2} \end{aligned}$$

Suppose that the area $\mathbb{A}_E(f(\mathbb{D}))$ is finite. Then $A(t)$ increases to its limiting value $\mathbb{A}_E(f(\mathbb{D}))$. So the integral

$$\int_0^{\rho_o} \frac{1}{\tanh^2 \frac{1}{2}\delta} \left(\frac{A(t + \delta) - A(t - \delta)}{4\pi} \right) dt$$

is bounded independently of ρ_o . Hence $L_E(\rho_o) = O(\rho_o^{1/2})$. More carefully, we can apply the above result to the arc from some value u_o up to ρ_o . The integrand $A(t + \delta) - A(t - \delta)$ is then no more than $\mathbb{A}_E(f(\mathbb{D})) - A(u_o - \delta)$. This tends to 0 as $u_o \nearrow \infty$, so we see that $L_E(\rho) = o(\rho_o^{1/2})$ as $\rho_o \rightarrow \infty$. Thus we have reproved Keogh's Theorem as in [BC].

Theorem 3.1 (Keogh)

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic map with $\mathbb{A}_E(f(\mathbb{D}))$ finite. For any argument θ , the Euclidean length $L_E(\rho_o)$ of the image under f of the radial arc $[0, r_o e^{i\theta}]$ of hyperbolic length ρ_o (so $r_o = \tanh \frac{1}{2}\rho_o$) satisfies

$$L_E(\rho_o) = o(\rho_o^{1/2}) \quad \text{as } \rho_o \rightarrow \infty.$$

□

Note that the proof given above only required the application of Proposition 2.1 to functions where the area of the image is small, for we knew that $A(t) \nearrow \mathbb{A}_E(f(\mathbb{D}))$ as $t \nearrow \infty$. Hence Propositions 2.2 and 2.3 give the corresponding results for maps into the hyperbolic plane and the Riemann sphere.

Theorem 3.2

Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic map with $\mathbb{A}_H(f(\mathbb{D}))$ finite. For any argument θ , the hyperbolic length $L_H(\rho_o)$ of the image under f of the radial arc $[0, r_o e^{i\theta}]$ of hyperbolic length ρ_o satisfies

$$L_H(\rho_o) = o(\rho_o^{1/2}) \quad \text{as } \rho_o \rightarrow \infty.$$

Theorem 3.3

Let $f : \mathbb{D} \rightarrow \mathbb{P}$ be an analytic map with $\mathbb{A}_S(f(\mathbb{D}))$ finite. For any argument θ , the spherical length $L_S(\rho_o)$ of the image under f of the radial arc $[0, r_o e^{i\theta}]$ of hyperbolic length ρ_o satisfies

$$L_S(\rho_o) = o(\rho_o^{1/2}) \quad \text{as } \rho_o \rightarrow \infty.$$

In [BC] examples were constructed of functions $f : \mathbb{D} \rightarrow \mathbb{C}$ showing that the power $\frac{1}{2}$ in Theorem 2.1 is the best possible. Since these examples had f bounded and the hyperbolic, Euclidean and spherical metrics are Lipschitz equivalent on any compact region inside unit disc, these examples also show that the power is best possible in Theorems 3.2 and 3.3.

It is easy to adapt the argument used above to more general situations. Suppose, for example, that the area $A(t)$ grows to infinity but we have control on the rate of growth. Then we can use the same ideas to obtain a bound $L(\rho_o) = O(\rho_o^\alpha)$ for suitable exponents α .

In more detail, fix $\alpha > 1$. A different splitting of the integrands in the appeal to the Cauchy – Schwarz inequality above gives us

$$\begin{aligned} L_E(\rho_o) &\leq \int_0^{\rho_o} \frac{1}{\tanh \frac{1}{2}\delta} \left(\frac{A(t+\delta) - A(t-\delta)}{4\pi} \right)^{1/2} dt \\ &\leq \left(\int_0^{\rho_o} \frac{1}{4\pi t^{\alpha-1} \tanh^2 \frac{1}{2}\delta} (A(t+\delta) - A(t-\delta)) dt \right)^{1/2} \left(\int_0^{\rho_o} t^{\alpha-1} dt \right)^{1/2} \\ &\leq \left(\int_0^{\rho_o} \frac{1}{4\pi t^{\alpha-1} \tanh^2 \frac{1}{2}\delta} (A(t+\delta) - A(t-\delta)) dt \right)^{1/2} \left(\frac{\rho_o^\alpha}{\alpha} \right)^{1/2} \end{aligned}$$

For the first integrand, we can ignore the behaviour near 0 and write

$$\begin{aligned} \int_0^{\rho_o} \frac{1}{t^{\alpha-1} \tanh^2 \frac{1}{2}\delta} (A(t+\delta) - A(t-\delta)) dt &= \\ &= \int^{\rho_o-\delta}_{\rho_0-\delta} \frac{1}{(t-\delta)^{\alpha-1} \tanh^2 \frac{1}{2}\delta} A(t) dt - \int^{\rho_o+\delta}_{\rho_0+\delta} \frac{1}{(t+\delta)^{\alpha-1} \tanh^2 \frac{1}{2}\delta} A(t) dt. \end{aligned}$$

So we want the integral

$$\int^{\infty} \left(\frac{1}{(t-\delta)^{\alpha-1}} - \frac{1}{(t+\delta)^{\alpha-1}} \right) \frac{A(t)}{\tanh^2 \frac{1}{2} \delta} dt$$

to converge. Then it will follow that $L_E(\rho_o) = o(\rho_o^{\alpha/2})$ as $\rho_o \nearrow \infty$.

The mean value theorem immediately gives

$$\frac{1}{(t-\delta)^{\alpha-1}} - \frac{1}{(t+\delta)^{\alpha-1}} \leq 2\delta(\alpha-1)(t-\delta)^{-\alpha} .$$

So we see that we require the area $A(t)$ to grow sufficiently slowly that

$$\int^{\infty} \frac{\delta}{\tanh \frac{1}{2} \delta} \frac{A(t)}{(t-\delta)^{\alpha}} dt$$

converges.

Using Proposition 2.2 in place of Proposition 2.1 gives us the same results for functions into the unit disc. In order to use Proposition 2.3 to obtain corresponding results for meromorphic functions into the Riemann sphere we need to ensure that $A(t+\delta) - A(t-\delta)$ is sufficiently small for 2.3 to apply. In order to achieve this the radius δ usually needs to decrease as t increases.

The natural class of functions to consider here is those of finite order, so the Nevanlinna characteristic $T(r)$ satisfies

$$\limsup_{r \rightarrow 1} \frac{T(r)}{\log \frac{1}{1-r}} < \infty .$$

(See [T].) This implies that

$$\int^{\infty} T(r)(1-r)^{k-1} dr < \infty$$

for some k . The derivative $T'(r)$ is $S(r)/r$ where

$$S(r) = \frac{\mathbb{A}_S(f(\{z : |z| < r\}))}{4\pi} = \frac{A_S(t)}{4\pi}$$

for $t = \log(1+r)/(1-r)$, which is the hyperbolic radius corresponding to the Euclidean radius r . So, integrating by parts gives

$$\int^{\infty} A_S(t) \exp(-(k+1)t) dt < \infty .$$

In order to obtain estimates for these functions we would need to take the radius δ tending exponentially to 0 as t increased to ∞ . The details do not seem inspiring.

4. Examples

As in [BC], it is useful to consider examples that limit what can happen to the lengths of the images of arcs. As there we can construct many examples defined on a region

$$S = \{x + iy \in \mathbb{C} : |y| < h(x)\}$$

for some slowly increasing function h . This will be a hyperbolic simply-connected domain, so it is conformally equivalent to the unit disc. Let $q : \mathbb{D} \rightarrow S$ be the conformal map which fixes the origin. The hyperbolic metric is Lipschitz equivalent to the pseudo-hyperbolic metric which has density $1/d_E(z, \partial S)$. The positive real axis is then a hyperbolic geodesic in S and the length of the segment from the origin to t_o is approximately

$$\rho_o = \int_0^{t_o} \frac{1}{h(t)} dt .$$

The hyperbolic disc $B(0, \rho_o)$ is then certainly contained in the part of S to the left of $\{x + iy \in S : y < t_o\}$.

In [BC] the function q itself was considered. For us, we need to follow q by a mapping that is an isometry from the positive real axis with the Euclidean metric to the hyperbolic plane or the Riemann sphere with their metrics. For this we follow q by an exponential map. This gives us corresponding examples for maps into the hyperbolic plane or the Riemann sphere.

It may also be worth considering analogous examples for maps into the Riemann sphere where we constrain the Nevanlinna characteristic rather than the spherical area of the image. If the analytic map $f : \mathbb{D} \rightarrow \mathbb{P}$ has $\mathbb{A}(f(\mathbb{D}))$ finite, then we certainly have

$$T(r) = T(\tfrac{1}{2}) + \int_{\frac{1}{2}}^r \frac{\mathbb{A}(f(\mathbb{D}))}{4\pi s} ds = T(\tfrac{1}{2}) + \frac{\mathbb{A}(f(\mathbb{D}))}{4\pi} \log 2r .$$

So the Nevanlinna characteristic is bounded. However, we do not have $L_S(\rho_o) = o(\rho_o^{1/2})$ for every function f with bounded characteristic.

Consider the universal cover of an annulus $\{z \in \mathbb{C} : R^{-1} < |z| < R\}$, say $q : \mathbb{R}_+^2 \rightarrow \{z \in \mathbb{C} : R^{-1} < |z| < R\}$ defined on the upper half-plane \mathbb{R}_+^2 . We can arrange for the hyperbolic geodesic $\{iy : y > 0\}$ to be mapped to the unit circle. If the segment from i to Ki is mapped to one complete circuit of the circle, then so are the segments from $K^n i$ to $K^{n+1} i$ for every integer n . Consequently we see that the hyperbolic geodesic from i to $i\infty$ has an image with $L_S(\rho_o) \sim \rho_o$ and not a half power.

An almost identical argument gives the same conclusion for the Blaschke product

$$B(z) = \prod_{n=-\infty}^{-1} (-1) \left(\frac{2^n i - z}{2^n i + z} \right) \prod_{n=0}^{\infty} \left(\frac{2^n i - z}{2^n i + z} \right)$$

with zeros evenly spaced hyperbolically along the imaginary axis. The image of the positive imaginary axis is now the curve that traces out repeatedly the line segment between the two critical values of B .

This shows that we can not hope for a better inequality than $L_S(\rho_o) = O(\rho_o)$ for functions with bounded Nevanlinna characteristic. Even this is untrue, as the following example shows.

Let (y_n) be a strictly increasing sequence of strictly positive real numbers with $\sum 1/y_n$ convergent. Then there is a Blaschke product

$$B(z) = \prod \left(\frac{iy_n - z}{iy_n + z} \right)$$

with zeros at the points iy_n . This product is symmetric about the imaginary axis with

$$B(-\bar{z}) = \overline{B(z)} .$$

So, between any two successive zeros iy_n and iy_{n+1} , there is a single critical point. We will be interested in the case where the y_n converge slowly to ∞ , so the hyperbolic distances $d_H(iy_n, iy_{n+1}) = \log y_{n+1}/y_n$ decrease to 0 as $n \rightarrow \infty$. The Blaschke product is a contraction for the hyperbolic metric because of the Schwarz – Pick lemma, so the images under f of the critical points must converge to 0. The image of the positive imaginary axis then traces out line segments on the real axis between successive critical values and converges to 0. Now consider the geodesic $\gamma(t) = -1 + ie^t$ in the upper half-plane. As $t \rightarrow \infty$, this becomes closer, in the hyperbolic metric, to the positive imaginary axis. Hence we see that image $B(\gamma)$ traces out a path that never takes the value 0 and winds (negatively) about 0. Since $d_H(\gamma(t), ie^t) < 1/e^t$, we see that, for large n , the path $B(\gamma[iy_n, iy_{n+1}])$ completes approximately half a circuit about 0.

Define $f : \mathbb{R}_+^2 \rightarrow \mathbb{P}$ by

$$f(z) = \frac{B(z+1)}{B(z-1)} .$$

This is an analytic function with bounded Nevanlinna characteristic because of Fatou's theorem, which says that the ratio of two bounded analytic functions has bounded characteristic (see [N] or Proposition 4.1 later). For a point iy on the imaginary axis we have

$$f(iy) = \frac{B(1+iy)}{B(-1+iy)} = \frac{B(1+iy)}{\overline{B(1+iy)}} .$$

So the path $f(\gamma[iy_n, iy_{n+1}])$ lies on the unit circle and completes approximately one circuit about 0 for each sufficiently large n . If we set $\rho_o = \log y_n$ to be the hyperbolic distance from i to iy_n , then we have

$$L_S(\rho_o) = L_S(\log y_n) \sim 2\pi n .$$

For example, take $y_n = n^2$. Then

$$L_S(\rho_o) \sim 2\pi \exp \frac{1}{2}\rho_o .$$

So we certainly do not have $L_S(\rho_o) = O(\rho_o)$.

The issue here is that the Nevanlinna characteristic depends crucially on the choice of the origin. When we try to apply the arguments of Theorem 3.3 we take each point of the radial arc as the origin and so we require a bound on the Nevanlinna characteristic independent of the position of the origin. We will see that such a stronger, uniform condition is enough to give $L_P(\rho_o) = O(\rho_o)$. It will be useful in this context to have a precise form of Fatou's Theorem:

Proposition 4.1 Fatou

For a meromorphic function $f : \mathbb{D} \rightarrow \mathbb{P}$ the following conditions are equivalent.

- (a) *f has bounded characteristic with $T(1) \leq K$;*
- (b) *There are two analytic functions $f_0, f_\infty : \mathbb{D} \rightarrow \mathbb{C}$ with $f = f_0/f_\infty$,*

$$|f_0(z)|^2 + |f_\infty(z)|^2 \leq 1 \quad \text{and} \quad |f_0(0)|^2 + |f_\infty(0)|^2 \geq e^{-2K}$$

for all $z \in \mathbb{D}$.

Proof:

First we will consider this result when the function f extends analytically across the boundary \mathbb{D} with no zeros or poles on $\{z : |z| = 1\}$. We can then obtain the general result by applying this to the functions restricted to discs of radius $r < 1$.

We will need to use the chordal distance $k(w, w')$ on the Riemann sphere. This is the length of the chord in \mathbb{R}^3 joining the points $w, w' \in \mathbb{P}$, so

$$k(w, w') = \frac{2|w - w'|}{\sqrt{1 + |w|^2}\sqrt{1 + |w'|^2}} \quad \text{and} \quad k(w, w') = 2 \sin \frac{1}{2} d_S(w, w') .$$

Write $f(z) = \frac{B_0(z)}{B_\infty(z)} \exp h(z)$ where B_0, B_∞ are finite Blaschke products on the zeros and poles of f and h is analytic on the closed unit disc. Let u_0, u_∞ be continuous functions on the closed disc, harmonic on the interior and with boundary values

$$u_0(\zeta) = \log k(f(\zeta), 0) , \quad u_\infty(\zeta) = \log k(f(\zeta), \infty) \quad \text{for } |\zeta| = 1. \quad (4.1)$$

Then

$$u_0(\zeta) - u_\infty(\zeta) = \log \frac{k(f(\zeta), 0)}{k(f(\zeta), \infty)} = \log |f(\zeta)| = \operatorname{Re} h(\zeta)$$

and so $u_0(z) - u_\infty(z) = \operatorname{Re} h(z)$ for each $z \in \mathbb{D}$. We can choose harmonic conjugates \tilde{u}_0 and \tilde{u}_∞ with $(u_0 + i\tilde{u}_0) - (u_\infty + i\tilde{u}_\infty) = h$. Now set

$$f_0 = \frac{1}{2}B_0 \exp(u_0 + i\tilde{u}_0) \quad \text{and} \quad f_\infty = \frac{1}{2}B_\infty \exp(u_\infty + i\tilde{u}_\infty).$$

This certainly gives $f = f_0/f_\infty$. For $|\zeta| = 1$ we have

$$\begin{aligned} |f_0(\zeta)|^2 + |f_\infty(\zeta)|^2 &= \frac{1}{4} \exp 2u_0(\zeta) + \frac{1}{4} \exp 2u_\infty(\zeta) \\ &= \frac{1}{4} (k(f(\zeta), 0)^2 + k(f(\zeta), \infty)^2). \end{aligned}$$

The points $0, \infty$ and $f(z)$ are the vertices of a right-angle triangle in \mathbb{P} , so Pythagoras' theorem shows that $k(f(z), 0)^2 + k(f(z), \infty)^2 = 2^2$. Hence

$$|f_0(\zeta)|^2 + |f_\infty(\zeta)|^2 = 1 \quad \text{for } |\zeta| = 1.$$

The First Nevanlinna Theorem shows that

$$T(1) = N(1; 0) + m(1; 0) = -\log |B_0(0)| + \int_0^{2\pi} \log \frac{k(f(0), 0)}{k(f(e^{i\theta}), 0)} \frac{d\theta}{2\pi}$$

and, since u_0 is harmonic, this is

$$T(1) = -\log |B_0(0)| - u_0(0) + \log k(f(0), 0) = -\log 2|f_0(0)| + \log k(f(0), 0).$$

Therefore, $|f_0(0)| = \frac{1}{2}k(f(0), 0) \exp -T(1)$ and, similarly, $|f_\infty(0)| = \frac{1}{2}k(f(0), \infty) \exp -T(1)$. Hence,

$$|f_0(0)|^2 + |f_\infty(0)|^2 = \frac{1}{4} (k(f(0), 0)^2 + k(f(0), \infty)^2) \exp -2T(1) = \exp -2T(1).$$

This shows that (a) implies (b). It is a little simpler to reverse this argument to prove that (b) implies (a).

Finally, for any analytic function $f : \mathbb{D} \rightarrow \mathbb{P}$ we can apply the above result to

$$f_r : \mathbb{D} \rightarrow \mathbb{P} ; \quad z \mapsto f(rz)$$

for those $r < 1$ with no zeros or poles of f on $\{z : |z| = r\}$. By taking locally uniform limits as $r \nearrow 1$ we obtain the proposition in general. \square

We will say that an analytic function $f : \mathbb{D} \rightarrow \mathbb{P}$ has *uniformly bounded characteristic* if there is a constant C with each of the functions

$$f_{z_o} : z \mapsto f\left(\frac{z + z_o}{1 + \overline{z_o}z}\right)$$

having characteristic $T(f_{z_o}; r)$ at most C for every $r < 1$. This is saying that the Nevanlinna characteristic is bounded independently of the origin z_o we choose in \mathbb{D} . The smallest value for C is clearly invariant under composing f with hyperbolic isometries of \mathbb{D} . Fatou's theorem immediately gives us:

Corollary 4.2

A meromorphic function $f : \mathbb{D} \rightarrow \mathbb{P}$ has uniformly bounded characteristic if and only if $f = f_0/f_\infty$ for two bounded analytic functions f_0, f_∞ with

$$\delta < (|f_0|^2 + |f_\infty|^2)^{1/2} \leq 1$$

for some $\delta > 0$.

This means that the pair of functions f_0, f_∞ are Corona data.

Proof:

Suppose that $f : \mathbb{D} \rightarrow \mathbb{P}$ has bounded characteristic. Then we can write $f = f_0/f_\infty$ where

$$f_0 = \frac{1}{2}B_0 \exp(u_0 + i\tilde{u}_0) \quad ; \quad f_\infty = \frac{1}{2}B_\infty \exp(u_\infty + i\tilde{u}_\infty)$$

and u_0, u_∞ are the harmonic functions with boundary values (4.1). Theorem 4.1 shows that

$$\exp -2T(f_{z_o}; 1) = |f_0(z_o)|^2 + |f_\infty(z_o)|^2.$$

Hence, $|f_0(z_o)|^2 + |f_\infty(z_o)|^2 \geq \delta^2$ if and only if $T(f_{z_o}; 1) \leq -\log \delta$. \square

We can now prove that the inequality $L_S(\rho_o) = O(\rho_o)$ does hold for functions f with uniformly bounded Nevanlinna characteristic.

Theorem 4.3

Let $f : \mathbb{D} \rightarrow \mathbb{P}$ be an analytic function with uniformly bounded Nevanlinna characteristic. Then

$$L_S(\rho_o) = O(\rho_o) .$$

Proof:

The corollary shows that we can write f as f_0/f_∞ where f_0, f_∞ are both bounded analytic functions. Moreover the function

$$F : \mathbb{D} \rightarrow \mathbb{C}^2 \quad ; \quad z \mapsto (f_0(z), f_\infty(z))$$

will satisfy $\delta \leq \|F(z)\| \leq 1$ for the Euclidean norm $\|\cdot\|$ on \mathbb{C}^2 .

The Schwarz – Pick lemma, applied to f_0 and f_∞ , shows that

$$\|F'(z)\|(1 - |z|^2) \leq 2 .$$

Also, a simple calculation gives

$$\begin{aligned} \|f'(z)\|_{H \rightarrow S} &= \left(\frac{2|f'(z)|}{1 + |f(z)|^2} \right) \left(\frac{1 - |z|^2}{2} \right) = \frac{|f'_0(z)f_\infty(z) - f_0(z)f'_\infty(z)|}{|f_0(z)|^2 + |f_\infty(z)|^2} (1 - |z|^2) \\ &\leq \frac{\|F'(z)\| \|F(z)\|}{\|F(z)\|^2} (1 - |z|^2) \leq \frac{\|F'(z)\|}{\|F(z)\|} (1 - |z|^2) . \end{aligned}$$

So we have

$$\|f'(z)\|_{H \rightarrow S} \leq \frac{2}{\|F(z)\|} \leq \frac{2}{\delta} .$$

Integrating this along a hyperbolic geodesic give the result. \square

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